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COMPARATIVE ANALYSIS OF THE INVERSE WEIBULL DISTRIBUTION BY USING TYPE-II CENSORED SAMPLE

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ABSTRACT

Parameter estimation become complicated when censoring is present in the sample. Some time it is not possible to give a mathematical expression about estimated values of parameters in Maximum Likelihood (ML) method. In this situation iteration method is used, to find estimated values of parameters in numeric form . There are several Modified Maximum Likelihood (MML) estimation procedures which provide a mathematical expression about parametric value. For the two-parameter inverse Weibull distribution by using type II censored sample, assuming fixed shape parameter, a modified maximum likelihood estimator is proposed by [1] in which they use a simple approximation for intractable terms to estimate the scale parameter. In this paper a comparative study of the proposed estimator [1] is made with MML estimator of [2] ML estimators of [3].

Keywords: Inverse weibull distribution; Type II censored sample; Type II censoring; Modified maximum likelihood estimator; order statistics; Bias; Asymptotic variance; Mean square error.

1. INTRODUCTION

The probability density function of inverse Weibull distribution is given as:

$$f(y)=\frac{m}{\theta}\frac{1}{y^{m+1}}\exp\left(-\frac{1}{\theta\, y^{m}}\right) \qquad , y>0,\ m,\ \theta>0. \tag{1.1}$$

= 0 other wise .

and the corresponding distribution function is:

$$F(y)=\exp\left(-\frac{1}{\theta\, y^{m}}\right) \tag{1.2}$$

Where "m" is shape and "θ" is scale parameter

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2. THE MODIFIED MAXIMUM LIKELIHOODESTIMATOR (MMLE) OF THE SCALE PARAMETER OF THE INVERSE WEIBULL DISTRIBUTION

For the doubly type ll censored sample with r samples censored on the left and s samples censored on the right .The likelihood function is given as

$$L = \frac{n!}{r!s!}\left[F(Y_{r+1})\right]^r \left[1 - F(Y_{n-s})\right]^s \prod_{i=r+1}^{n-s} f(y_i) \tag{2.1}$$

Where $r = [nq_1] + 1$ and $s = [nq_2] + 1$, q_1 is the proportion of left censored sample and q_2 is the proportion of right censored sample .By using (1.1)and(1.2) in (2.1) we get

$$L = \frac{n!}{r!s!}\left[\left(\exp\left(-\frac{1}{\theta\, y_{r+1}{}^{m}}\right)\right)\right]^r \left[1 - \exp\left(-\frac{1}{\theta\, y_{n-s}{}^{m}}\right)\right]^s \prod_{i=r+1}^{n-s}\left[\frac{m}{\theta}\frac{1}{y_i^{m+1}}\exp\left(-\frac{1}{\theta\, y_i{}^{m}}\right)\right]$$

The first derivatives of the log-likelihood function with respect to θ is given by

$$\frac{\partial \ln L}{\partial \theta} = r\left(\frac{1}{\theta^2\, y_{r+1}{}^{m}}\right) + \frac{s}{\theta^2\, y_{n-s}{}^{m}}\left[-\frac{\exp\left(-\frac{1}{\theta\, y_{n-s}{}^{m}}\right)}{1-\exp\left(-\frac{1}{\theta\, y_{n-s}{}^{m}}\right)}\right] - \frac{(n-s-r)}{\theta} + \frac{1}{\theta^2}\sum_{i=r+1}^{n-s}\frac{1}{y^{m}{}_{i}} \tag{2.2}$$

Putting $z_{n-s} = \theta y^{m}{}_{n-s}$ in equation (2.2) we get

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\theta^2 y^{m}{}_{r+1}} + \frac{s}{\theta^2 y^{m}{}_{n-s}}\left(\frac{-\exp\left(-\frac{1}{z_{n-s}}\right)}{1-\exp\left(-\frac{1}{z_{n-s}}\right)}\right) - \frac{(n-s-r)}{\theta} + \frac{1}{\theta^2}\sum_{i=r+1}^{n-s}\frac{1}{y^{m}{}_{i}} \overset{(set)}{=} 0 \tag{2.3}$$

By using $\exp(z^{-1}) = \frac{1+\frac{1}{2z}}{1-\frac{1}{2z}}$ for intractable term in (2.3), see [4]. Solving (2.3) for θ ,the modified maximum likelihood estimator of θ is given as

$$\theta^{\wedge} = \frac{\frac{r}{y_{r+1}{}^{m}} + \frac{s}{2y^{m}{}_{n-s}} + \sum_{i=r+1}^{n-s}\frac{1}{y^{m}{}_{i}}}{n-r} \tag{2.4}$$

3. ASYMPTOTIC VARIANCE AND BIAS OF $\theta^{\wedge}$

By using Glivenko-Cantelli lemma given by [5],
We have

$p_1 = G^{-1}(q_1)$ and $p_2 = G^{-1}(1-q_2)$ So as $n \to \infty$

$z_{r+1} = G^{-1}(q_1)$ and $z_{n-s} = G^{-1}(1-q_2)$

z_{r+1} can be evaluated by using following relation

$$G(z_{r+1}) = q_1$$

$$\int_0^{z_{r+1}} f(z)dz = q_1$$

$$z_{r+1} = -\frac{1}{\ln q1} \tag{3.1}$$

and

$$G(z_{n-s}) = 1 - q_2$$

$$\int_{z_{n-s}}^{\infty} f(z)dz = q_2$$

$$z_{n-s} = -\frac{1}{\ln(1-q_2)} \tag{3.2}$$

and

$$\lim_{n\to\infty} E\left\{\frac{1}{n}\sum_{i=r+1}^{n-s}\frac{1}{z_i}\right\} = \int_{z_{r+1}}^{z_{n-s}} \frac{1}{z} f(z)dz = \int_{z_{r+1}}^{z_{n-s}} \frac{1}{z^3}\exp\left(-\frac{1}{z}\right)dz$$

$$= 1 - \ln(1-q_2) - q_2 + q_2\ln(1-q_2) - q_1 + q_1\ln q_1 \tag{3.3}$$

From the equation (2.2) we have

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\theta^2 y^m{}_{r+1}} + \frac{s}{2\theta^2 y^m{}_{n-s}} - \frac{s}{\theta} - \frac{(n-s-r)}{\theta} + \frac{\sum_{i=r+1}^{n-s}\frac{1}{y^m{}_i}}{\theta^2}$$

Partially differentiate the above equation w.r.t θ

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{2r}{\theta^3 y^m{}_{r+1}} - \frac{2s}{2\theta^3 y^m{}_{n-s}} + \frac{s}{\theta^2} + \frac{(n-s-r)}{\theta^2} - \frac{2\sum_{i=r+1}^{n-s}\frac{1}{y^m{}_i}}{\theta^3}$$

Putting $z_{r+1} = y^m{}_{r+1}\theta, z_{n-s} = y^m{}_{n-s}\theta, z_i = y^m{}_i\theta$,we have

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{2r}{z_{r+1}\theta^2} - \frac{s}{z_{n-s}\theta^2} + \frac{s}{\theta^2} + \frac{(n-s-r)}{\theta^2} - \frac{2\sum_{i=r+1}^{n-s}\frac{1}{z_i}}{\theta^2} \tag{3.4}$$

By using equation (3.1) and (3.2) we obtain as:

$$-E\left(\frac{\partial^2 \ln L}{\partial\theta^2}\right)=\frac{n}{\theta^2}\left(-2q_1\ln q_1-q_2\ln(1-q_2)-1+q_1+2E\left(\frac{1}{n}\sum_{i=r+1}^{n-s}\frac{1}{z_i}\right)\right)$$

From equation (3.3)

$$=\frac{n}{\theta^2}\left[-2q_1\ln q_1-q_2\ln(1-q_2)-1+q_1+2-2\ln(1-q_2)-2q_2+2q_2\ln(1-q_2)-2q_1+2q_1\ln q_1\right]$$

So , Asymptotic variance is given as

$$\operatorname{var}(\theta^{\wedge})=\frac{\theta^2}{n[1-q_1-2q_2-2\ln(1-q_2)+q_2\ln(1-q_2)]} \tag{3.5}$$

Where $\therefore q_1=\frac{r}{n}\therefore q_2=\frac{s}{n}$

From (3.5) we observe as $n\to\infty,\operatorname{var}(\theta^{\wedge})\to 0$

Now put $z_{r+1}=y^m{}_{r+1}\theta, z_{n-s}=y^m{}_{n-s}\theta$ and $z_i=y^m{}_i\theta$ in the equation (2.4) ,we get

$$\theta^{\wedge}=\frac{\frac{\theta r}{z_{r+1}}+\frac{s\theta}{2z_{n-s}}+\theta\sum_{i=r+1}^{n-s}\frac{1}{z_i}}{n-r}$$

Apply expectation on both sides and from the equation (3.1), (3.2) and (3.3) we have

$$Bias=E(\theta^{\wedge})-\theta=\frac{\theta\left(\frac{1}{2}q_2\ln(1-q_2)-\ln(1-q_1)-q_2\right)}{1-q_1} \tag{3.6}$$

From (3.6) we observe as sample size increases, The Bias decreases i.e. $Bias\to 0$ for large sample size

4. ESTIMATION OF THE MEAN SQUARE ERROR

From [6] for moments of order statistics from inverse weibull distribution, we have

$\mu_{r:n}^{(k-m)}=E\left(y_{r:n}^{k-m}\right)$ then

$$\mu_{r:n}^{(k-m)}=\frac{C_{r:n}\Gamma\left(2-\frac{k}{m}\right)}{\theta^{\frac{k}{m}-1}}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j(a_j)^{\frac{k}{m}-2} \tag{4.1}$$

Where $C_{r:n}=\frac{n!}{(r-1)!(n-r)!}$

Putting k=0 in (4.1) we obtain

$$\mu_{r:n}^{(-m)}=E\left(y_{r:n}^{-m}\right)$$

$$E\left(\frac{1}{y_{r:n}^{m}}\right)=\theta\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{j}\left(\frac{1}{r+j}\right)^{2} \quad (4.2)$$

Where $\alpha_r=\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{j}\left(\frac{1}{r+j}\right)^{2}$

And by putting k=-m in (4.1) ,we have

$$\mu_{r:n}^{(-m-m)}=E\left(y_{r:n}^{-2m}\right)=E\left(\frac{1}{y_{r:n}^{m}}\right)^{2}$$

$$E\left(\frac{1}{y_{r:n}^{m}}\right)^{2}=2\theta^{2}\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{j}\left(\frac{1}{r+j}\right)^{3} \quad (4.3)$$

By using (4.2 and (4.3),we have)

$$\mathrm{var}\left(\frac{1}{y^{m}{}_{r}}\right)=\theta^{2}\beta_{r} \quad (4.4)$$

Where $\beta_r=2\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{j}\left(\frac{1}{r+j}\right)^{3}-(\alpha_r)^{2}$

Applying expectation on equation (2.3) and then by using equation (4.2), we obtain

$$Bias\left(\theta^{\wedge}\right)=E\left(\theta^{\wedge}\right)-\theta=\frac{\theta\left(q_1\alpha_{r+1}+\frac{q_2}{2}\alpha_{n-s}+\frac{\alpha}{n}-1+q_1\right)}{1-q_1} \quad (4.5)$$

Where $\alpha=\sum_{i=r+1}^{n-s}\alpha_i$

Now applying variance on equation (2.3), then by using equation (4.4) we obtained

$$Var\left(\theta^{\wedge}\right)=\frac{\theta^{2}\left(q^{2}{}_{1}\beta_{r+1}+\frac{q^{2}{}_{2}}{4}\beta_{n-s}+\frac{\beta}{n^{2}}\right)}{(1-q_1)^{2}} \quad (4.6)$$

Where $\beta=\sum_{i=r+1}^{n-s}\beta_i$

By using equation (4.5) and (4.6),we have

$$MSE\left(\theta^{\wedge}\right)=\frac{\theta^{2}}{(1-q_1)^{2}}\left(\left(q_1\alpha_{r+1}+\frac{q_2}{2}\alpha_{n-s}+\frac{\alpha}{n}-1+q_1\right)^{2}+\left(q^{2}{}_{1}\beta_{r+1}+\frac{q^{2}{}_{2}}{4}\beta_{n-s}\frac{\beta}{n^{2}}\right)\right) \quad (4.7)$$

5. EMPIRICAL STUDY

5.1 The Modified Maximum Likelihoodestimator (MMLE) of mehrotra and Nanda

Mehrotra and Nanda [2] used the approach of MML, by replacing the hazard function in likelihood equation with its expectation in case of right Type ll censored sample of normal and gamma distribution. The reference [7] used the Mehrotra and Nanda [2] MML technique .They replace the intractable term of likelihood equations with its expectation, to get the estimators of location parameter by considering the scale parameter equal to of logistic distribution in right Type ll censored sample.

By using Mehrotra and Nanda [2] for intractable term in (2.3) the modified maximum likelihood estimator of θ is given as

$$\theta^{\wedge} = \frac{\dfrac{r}{y^{m}{}_{r+1}} + \dfrac{(s-n)}{y^{m}{}_{n-s}} + \sum_{i=r+1}^{n-s} \dfrac{1}{y^{m}{}_{i}}}{n-s-r} \tag{5.1}$$

And

$$E\left(\theta^{\wedge}\right) = \theta$$

And asymptotic Variance of $\theta^{\wedge}$ is given as

$$\operatorname{var}\left(\theta^{\wedge}\right) = \frac{\theta^2}{n\left(1-q_1-q_2\right)} \tag{5.2}$$

From (5.2) we observe as $n \to \infty$, $\operatorname{var}\left(\theta^{\wedge}\right) \to 0$

5.2 Estimation of the Mean Square Error

From (5.1) by using results in (4.2) and (4.4) we have

$$Bias\left(\theta^{\wedge}\right) = E\left(\theta^{\wedge}\right) - \theta = \frac{\theta\left(q_1\alpha_{r+1} + (q_2-1)\,\alpha_{n-s} + \dfrac{\alpha}{n} - 1 + q_1 + q_2\right)}{1-q_1-q_2} \tag{5.2.1}$$

$$\operatorname{var}\left(\theta^{\wedge}\right) = \frac{\theta^2\left(q_1{}^2\beta_{r+1} + \left(q_2-1\right)^2\beta_{n-s} + \dfrac{\beta}{n^2}\right)}{\left(1-q_1-q_2\right)^2} \tag{5.2.2}$$

From the equation (5.2.1) and (5.2.2) we obtain MSE as:

$$MSE(\theta) = \frac{\theta^2}{\left(1-q_1-q_2\right)^2}\left(\left(q_1\alpha_{r+1} + (q_2-1)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)^2 + \left(q_1{}^2\beta_{r+1} + \left(q_2-1\right)^2\beta_{n-s} + \frac{\beta}{n^2}\right)\right) \tag{5.2.3}$$

5.3 ML Estimator of Kambo

The explicit solution for ML estimator of two parameter exponential distribution in the case of doubly type ll censored sample.

In this section ML estimator are given for two parameters inverse weibull distribution in the case of doubly type ll censored sample by [3] approach.

By using [3] put (3.2) in (2.3) then the maximum likelihood estimator of θ is given as

$$\theta^{\wedge} = \frac{\left(\frac{r}{y^{m}{}_{r+1}}\right) - \frac{s}{y^{m}{}_{n-s}}\left(\frac{1-q_2}{q_2}\right) + \sum_{i=r+1}^{n-s}\frac{1}{y^{m}{}_{i}}}{n-r-s} \tag{5.4}$$

and $E(\theta^{\wedge}) = \theta$

and asymptotic Variance of $\theta^{\wedge}$ is given as

$$\operatorname{var}(\theta^{\wedge}) = \frac{\theta^2}{n[1-q_1-q_2]} \tag{5.5}$$

From (5.5) we observe as $n \to \infty, \operatorname{var}(\theta^{\wedge}) \to 0$

5.4 Estimation of the Mean Square Error

From (5.4) by using results in (4.2) and (4.4) we have

$$Bias(\theta^{\wedge}) = E(\theta^{\wedge}) - \theta = \frac{\left(q_1\alpha_{r+1} - (1-q_2)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)}{(1-q_1-q_2)} \tag{5.5.1}$$

$$\operatorname{var}(\theta^{\wedge}) = \frac{\left(q_1{}^2\beta_{r+1} + \beta_{n-s}(1-q_2)^2 + \frac{\beta}{n^2}\right)}{(1-q_1-q_2)^2} \tag{5.5.2}$$

From the equation (5.5.1) and (5.5.2) we obtain MSE as:

$$MSE(\theta) = \frac{\theta^2}{(1-q_1-q_2)^2}\left(\left(q_1\alpha_{r+1} + (1-q_2)\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1 + q_2\right)^2 + \left(q_1{}^2\beta_{r+1} + (1-q_2)^2\beta_{n-s} + \frac{\beta}{n^2}\right)\right) \tag{5.5.3}$$

Corollary: By substituting the value of q_2 in the equation (5.4) then the scale parameter obtained by [3] approach become scale parameter of [2].

Proof: Substitute the value of $q_2 = \frac{s}{n}$ in the equation (5.4),we get

$$\theta^{\wedge} = \frac{\left(\frac{r}{y^{m}{}_{r+1}}\right) - \frac{s}{y^{m}{}_{n-s}}\left(\frac{1-\frac{s}{n}}{\frac{s}{n}}\right) + \sum_{i=r+1}^{n-s}\frac{1}{y^{m}{}_{i}}}{n-r-s}$$

$$\theta^{\wedge} = \frac{\left(\frac{r}{y^{m}{}_{r+1}}\right) + \frac{(s-n)}{y^{m}{}_{n-s}} + \sum_{i=r+1}^{n-s} \frac{1}{y^{m}{}_{i}}}{n-r-s}$$

Equation (5.4) and (5.1) are same , this establishes Corollary.

Table 1. The asymptotic variance of MML estimates from doubly censored sample in term of θ^2 for n=10,20,30,50 and 100

q1	q2	n=10	n=10	n=10
		$\mathrm{var}(\theta^{\wedge})p/\theta^2$	$\mathrm{var}(\theta^{\wedge})k/\theta^2$	$\mathrm{var}(\theta^{\wedge})m\,\&\,n/\theta^2$
0	0	0.1	0.1	0.1
0	0.1	0.1	0.1111	0.1111
0	0.2	0.0998	0.125	0.125
0	0.3	0.0994	0.1429	0.1429
0	0.4	0.0983	0.1667	0.1667
0	0.5	0.0962	0.2	0.2
0	0.6	0.0924	0.25	0.25
0.1	0	0.1111	0.1111	0.1111
0.1	0.1	0.1111	0.125	0.125
0.1	0.2	0.1109	0.1429	0.1429
0.1	0.3	0.1103	0.1667	0.1667
0.1	0.4	0.109	0.2	0.2
0.1	0.5	0.1064	0.25	0.25
0.2	0	0.125	0.125	0.125
0.2	0.1	0.125	0.1429	0.1429
0.2	0.2	0.1247	0.1667	0.1667
0.2	0.3	0.124	0.2	0.2
0.2	0.4	0.1224	0.25	0.25
0.3	0	0.1429	0.1429	0.1429
0.3	0.1	0.1428	0.1667	0.1667
0.3	0.2	0.1425	0.2	0.2
0.3	0.3	0.1416	0.25	0.25

Where

$\mathrm{var}(\theta^{\wedge})p/\theta^2$ =Proposed variance from (3.5) $\mathrm{var}(\theta^{\wedge})k/\theta^2$ = variance by Kambo from (5.5) $\mathrm{var}(\theta^{\wedge})m\,\&\,n/\theta^2$ = variance by Mehrotra and Nanda from (5.2)

q1	q2	n=20	n=20	n=20
		$\mathrm{var}(\theta^{\wedge})p/\theta^2$	$\mathrm{var}(\theta^{\wedge})k/\theta^2$	$\mathrm{var}(\theta^{\wedge})m\,\&\,n/\theta^2$
0	0	0.05	0.05	0.05
0	0.1	0.05	0.0556	0.0556
0	0.2	0.0499	0.0625	0.0625
0	0.3	0.0497	0.0714	0.0714
0	0.4	0.0491	0.0833	0.0833
0	0.5	0.0481	0.1	0.1
0	0.6	0.0462	0.125	0.125
0.1	0	0.0556	0.0556	0.0556
0.1	0.1	0.0555	0.0625	0.0625
0.1	0.2	0.0555	0.0714	0.0714
0.1	0.3	0.0552	0.0833	0.0833
0.1	0.4	0.0545	0.1	0.1
0.1	0.5	0.0532	0.125	0.125
0.2	0	0.0625	0.0625	0.0625

q1	**q2**	**n=20**	**n=20**	**n=20**
		$\mathrm{var}(\theta^{\wedge})p/\theta^2$	$\mathrm{var}(\theta^{\wedge})k/\theta^2$	$\mathrm{var}(\theta^{\wedge})m\,\&\,n/\theta^2$
0.2	0.1	0.0625	0.0714	0.0714
0.2	0.2	0.0624	0.0833	0.0833
0.2	0.3	0.062	0.1	0.1
0.2	0.4	0.0612	0.125	0.125
0.3	0	0.0714	0.0714	0.0714
0.3	0.1	0.0714	0.0833	0.0833
0.3	0.2	0.0713	0.1	0.1
0.3	0.3	0.0708	0.125	0.125

q1	**q2**	**n=30**	**n=30**	**n=30**
		$\mathrm{var}(\theta^{\wedge})p/\theta^2$	$\mathrm{var}(\theta^{\wedge})k/\theta^2$	$\mathrm{var}(\theta^{\wedge})m\,\&\,n/\theta^2$
0	0	0.0333	0.0333	0.0333
0	0.1	0.0333	0.037	0.037
0	0.2	0.0333	0.0417	0.0417
0	0.3	0.0331	0.0476	0.0476
0	0.4	0.0328	0.0556	0.0556
0	0.5	0.0321	0.0667	0.0667
0	0.6	0.0308	0.0833	0.0833
0.1	0	0.037	0.037	0.037
0.1	0.1	0.037	0.0417	0.0417
0.1	0.2	0.037	0.0476	0.0476
0.1	0.3	0.0368	0.0556	0.0556
0.1	0.4	0.0363	0.0667	0.0667
0.1	0.5	0.0355	0.0833	0.0833
0.2	0	0.0417	0.0417	0.0417
0.2	0.1	0.0417	0.0476	0.0476
0.2	0.2	0.0416	0.0556	0.0556
0.2	0.3	0.0413	0.0667	0.0667
0.2	0.4	0.0408	0.0833	0.0833
0.3	0	0.0476	0.0476	0.0476
0.3	0.1	0.0476	0.0556	0.0556
0.3	0.2	0.0475	0.0667	0.0667
0.3	0.3	0.0472	0.0833	0.0833

q1	**q2**	**n=50**	**n=50**	**n=50**
		$\mathrm{var}(\theta^{\wedge})p/\theta^2$	$\mathrm{var}(\theta^{\wedge})k/\theta^2$	$\mathrm{var}(\theta^{\wedge})m\,\&\,n/\theta^2$
0	0	0.02	0.02	0.02
0	0.1	0.02	0.0222	0.0222
0	0.2	0.02	0.025	0.025
0	0.3	0.0199	0.0286	0.0286
0	0.4	0.0197	0.0333	0.0333
0	0.5	0.0192	0.04	0.04
0	0.6	0.0185	0.05	0.05
0.1	0	0.0222	0.0222	0.0222
0.1	0.1	0.0222	0.025	0.025
0.1	0.2	0.0222	0.0286	0.0286
0.1	0.3	0.0221	0.0333	0.0333
0.1	0.4	0.0218	0.04	0.04
0.1	0.5	0.0213	0.05	0.05
0.2	0	0.025	0.025	0.025
0.2	0.1	0.025	0.0286	0.0286
0.2	0.2	0.0249	0.0333	0.0333
0.2	0.3	0.0248	0.04	0.04

q1	q2	n=50	n=50	n=50
		$\mathrm{var}(\theta^{\wedge})p/\theta^2$	$\mathrm{var}(\theta^{\wedge})k/\theta^2$	$\mathrm{var}(\theta^{\wedge})m\&n/\theta^2$
0.2	0.4	0.0245	0.05	0.05
0.3	0	0.0286	0.0286	0.0286
0.3	0.1	0.0286	0.0333	0.0333
0.3	0.2	0.0285	0.04	0.04
0.3	0.3	0.0283	0.05	0.05

q1	q2	n=100	n=100	n=100
		$\mathrm{var}(\theta^{\wedge})p/\theta^2$	$\mathrm{var}(\theta^{\wedge})k/\theta^2$	$\mathrm{var}(\theta^{\wedge})m\&n/\theta^2$
0	0	0.01	0.01	0.01
0	0.1	0.01	0.0111	0.0111
0	0.2	0.01	0.0125	0.0125
0	0.3	0.0099	0.0143	0.0143
0	0.4	0.0098	0.0167	0.0167
0	0.5	0.0096	0.02	0.02
0	0.6	0.0092	0.025	0.025
0.1	0	0.0111	0.0111	0.0111
0.1	0.1	0.0111	0.0125	0.0125
0.1	0.2	0.0111	0.0143	0.0143
0.1	0.3	0.011	0.0167	0.0167
0.1	0.4	0.0109	0.02	0.02
0.1	0.5	0.0106	0.025	0.025
0.2	0	0.0125	0.0125	0.0125
0.2	0.1	0.0125	0.0143	0.0143
0.2	0.2	0.0125	0.0167	0.0167
0.2	0.3	0.0124	0.02	0.02
0.2	0.4	0.0122	0.025	0.025
0.3	0	0.0143	0.0143	0.0143
0.3	0.1	0.0143	0.0167	0.0167
0.3	0.2	0.0143	0.02	0.02
0.3	0.3	0.0142	0.025	0.025

Tables 2. The MSE of MML estimates from doubly censored sample in term of θ^2 for n=10,20and 30

q1	q2	n=10	n=10	n=10
		$MSE(\theta^{\wedge})p/\theta^2$	$MSE(\theta^{\wedge})k/\theta^2$	$MSE(\theta^{\wedge})m\&n/\theta^2$
0	0	0.0293	0.0493	0.0493
0	0.1	0.0292	0.0707	0.0707
0	0.2	0.0294	0.0989	0.0989
0	0.3	0.03	0.1371	0.1371
0	0.4	0.0317	0.1917	0.1917
0	0.5	0.0362	0.2747	0.2747
0	0.6	0.0473	0.4129	0.4129
0.1	0	0.0238	0.0485	0.0485
0.1	0.1	0.0238	0.0739	0.0739
0.1	0.2	0.0239	0.1087	0.1087
0.1	0.3	0.0246	0.1589	0.1589
0.1	0.4	0.0268	0.2361	0.2361
0.1	0.5	0.0323	0.3668	0.3668
0.2	0	0.0317	0.0629	0.0629
0.2	0.1	0.0316	0.0985	0.0985
0.2	0.2	0.0318	0.1507	0.1507
0.2	0.3	0.0327	0.2328	0.2328

q1	**q2**	**n=10** $MSE(\theta^{\wedge})p/\theta^2$	**n=10** $MSE(\theta^{\wedge})k/\theta^2$	**n=10** $MSE(\theta^{\wedge})m\&n/\theta^2$
0.2	0.4	0.0355	0.3751	0.3751
0.3	0	0.0455	0.0863	0.0863
0.3	0.1	0.0454	0.1397	0.1397
0.3	0.2	0.0456	0.225	0.225
0.3	0.3	0.0468	0.3761	0.3761

Where

$MSE(\theta^{\wedge})p/\theta^2$ =Proposed MSE from (4.7) $MSE(\theta^{\wedge})k/\theta^2$ =by Kambo from (5.5.3) $MSE(\theta^{\wedge})m\&n/\theta^2$ =by Mehrotra and Nanda from (5.2.3)

q1	**q2**	**n=20** $MSE(\theta^{\wedge})p/\theta^2$	**n=20** $MSE(\theta^{\wedge})k/\theta^2$	**n=20** $MSE(\theta^{\wedge})m\&n/\theta^2$
0	0	0.009	0.014	0.014
0	0.1	0.009	0.0225	0.0225
0	0.2	0.0091	0.0336	0.0336
0	0.3	0.0094	0.0484	0.0484
0	0.4	0.0104	0.0692	0.0692
0	0.5	0.0127	0.1003	0.1003
0	0.6	0.0186	0.1511	0.1511
0.1	0	0.0086	0.0148	0.0148
0.1	0.1	0.0086	0.0254	0.0254
0.1	0.2	0.0087	0.0397	0.0397
0.1	0.3	0.0091	0.0603	0.0603
0.1	0.4	0.0103	0.0916	0.0916
0.1	0.5	0.0132	0.1441	0.1441
0.2	0	0.014	0.0218	0.0218
0.2	0.1	0.014	0.0372	0.0372
0.2	0.2	0.0141	0.0596	0.0596
0.2	0.3	0.0147	0.0947	0.0947
0.2	0.4	0.0162	0.1556	0.1556
0.3	0	0.0219	0.0321	0.0321
0.3	0.1	0.0219	0.0555	0.0555
0.3	0.2	0.0221	0.0929	0.0929
0.3	0.3	0.0228	0.1591	0.1591

q1	**q2**	**n=30** $MSE(\theta^{\wedge})p/\theta^2$	**n=30** $MSE(\theta^{\wedge})k/\theta^2$	**n=30** $MSE(\theta^{\wedge})m\&n/\theta^2$
0	0	0.0044	0.0067	0.0067
0	0.1	0.0044	0.0118	0.0118
0	0.2	0.0045	0.0184	0.0184
0	0.3	0.0047	0.0272	0.0272
0	0.4	0.0054	0.0394	0.0394
0	0.5	0.0071	0.0575	0.0575
0	0.6	0.0114	0.0865	0.0865
0.1	0	0.005	0.0078	0.0078
0.1	0.1	0.005	0.0144	0.0144
0.1	0.2	0.0051	0.0233	0.0233
0.1	0.3	0.0054	0.036	0.036
0.1	0.4	0.0062	0.0553	0.0553
0.1	0.5	0.0083	0.0875	0.0875
0.2	0	0.009	0.0124	0.0124

q1	**q2**	**n=30** $MSE(\theta^{\wedge})p/\theta^2$	**n=30** $MSE(\theta^{\wedge})k/\theta^2$	**n=30** $MSE(\theta^{\wedge})m\,\&\,n/\theta^2$
0.2	0.1	0.009	0.0221	0.0221
0.2	0.2	0.0091	0.0363	0.0363
0.2	0.3	0.0095	0.0585	0.0585
0.2	0.4	0.0105	0.0968	0.0968
0.3	0	0.0144	0.019	0.019
0.3	0.1	0.0144	0.0338	0.0338
0.3	0.2	0.0146	0.0576	0.0576
0.3	0.3	0.0151	0.0997	0.0997

6. DISCUSSION AND CONCLUSION

1) The purpose of conducting empirical study is to see the closeness of MML estimators to ML estimator of Kambo .Since it has been proved that MML estimator of Mehrotra and Nanda are exactly same to Kambo estimator and they also equivalent in terms of asymptotic variance and MSE.

2) The asymptotic variance of Kambo, Mehrotra & Nanda and Proposed asymptotic variance are given in the tables (5.1) for n= 10 ,20 ,30 ,50 and 100.It can be observed that Proposed asymptotic variance is minimum as compared to the asymptotic variance of Kambo and Mehrotra & Nanda .The asymptotic variance of Kambo and Mehrotra & Nanda are exactly same.

3) For q_1=q_2=0 . no censoring scheme is involved in other words there is no missing element in the sample then the Proposed asymptotic variance ,asymptotic variance by Kambo and asymptotic variance by Mehrotra and Nanda are exactly same for any sample of size n.

4) The Mean square error of Kambo, Mehrotra & Nanda and Proposed MSE are given in the tables (5.2) for n=10 ,20 and 30.The MSE of scale parameter by Mehrotra & Nanda and ML estimator by Kambo are exactly same. It also observed that Proposed MSE are minimum as compared to the MSE's of Kambo and Mehrotra & Nanda.

5) As the sample size increases the Mean square errors and asymptotic variances of all MML and ML estimators tends to decreases .

6) Proposed Mean square error and asymptotic variance is minimum as compared to the Mean square error and asymptotic variance of Kambo and Mehrotra & Nanda respectively.

For further the authors refer [8-133], to recent advances in statistics for the readers.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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